The Angel versus The Devil

A mathematical chase

Stijn Vermeeren

University of Leeds

February 2, 2011

The game

- **An infinite chessboard**;

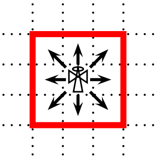
The game

- **An infinite chessboard**;
- **The Devil** (Quadraphage) eats a square on every turn;

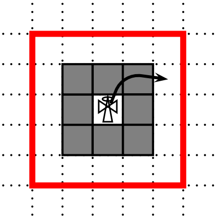
The game

- **An infinite chessboard**;
- **The Devil** (Quadraphage) eats a square on every turn;
- **A** k-**Angel** (Angel of power k) sits on a square, and on each turn flies to an uneaten square within l_∞ distance k.

1-Angel



2-Angel



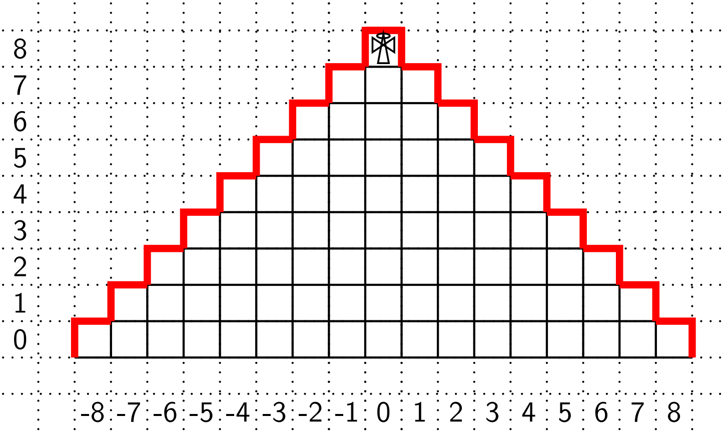
The aim of the game

- **The Devil** wants to trap the Angel (i.e. every square within reach of the Angel is eaten);
- **The Angel** wants to live forever.

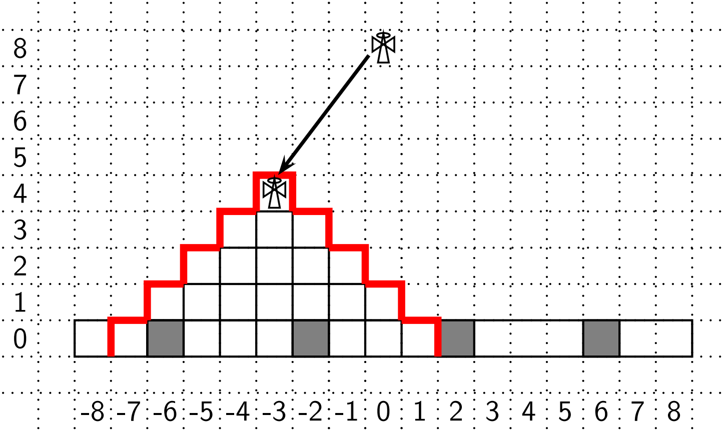
The Angel problem

- Game invented by David Silverman (1940s).
- Popularly known in 1970s (e.g. Martin Gardner).
- Berlekamp, Conway, Guy (1982):
  - 1-Angel loses.
  - **The Angel problem**:
    Can an Angel of sufficient power survive?

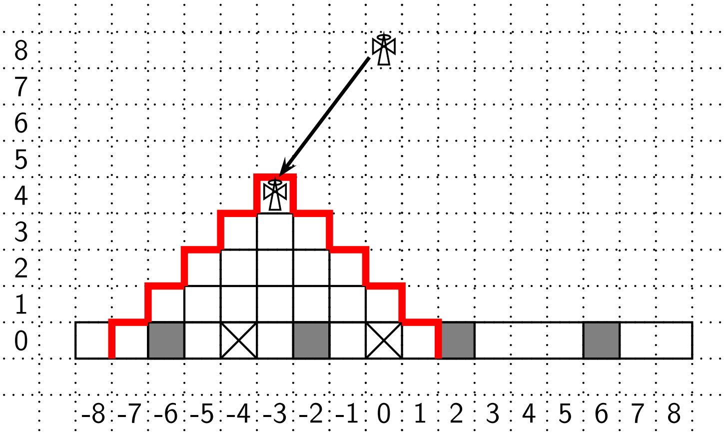
Catching a 1-Fool



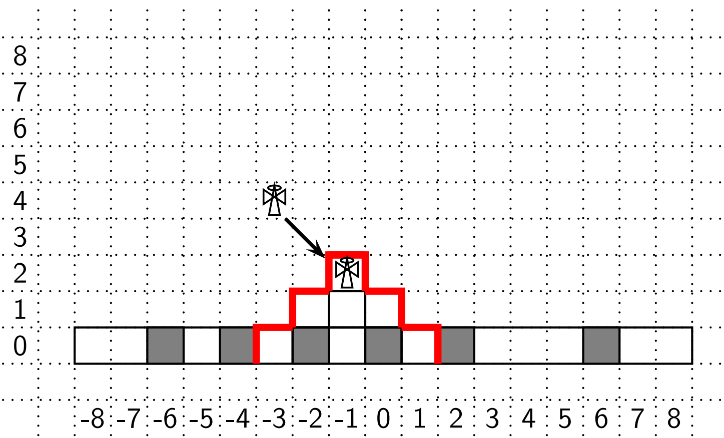
Catching a 1-Fool



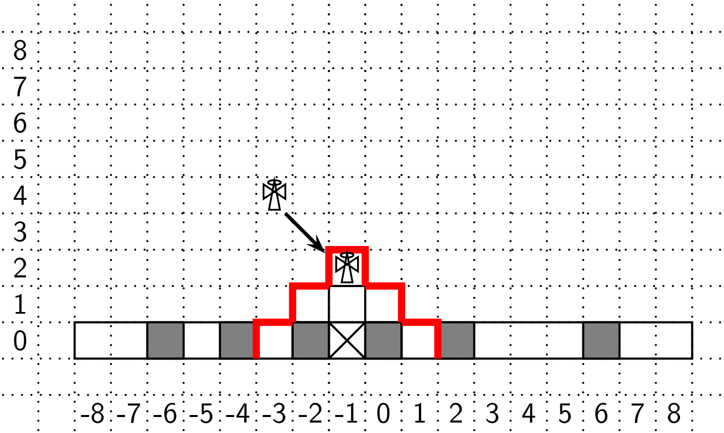
Catching a 1-Fool



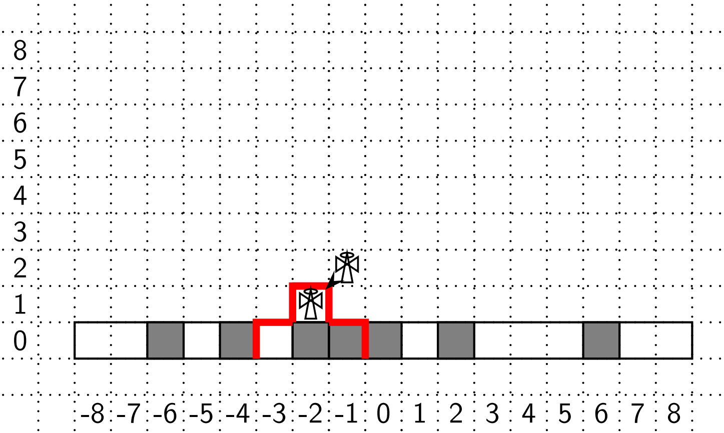
Catching a 1-Fool



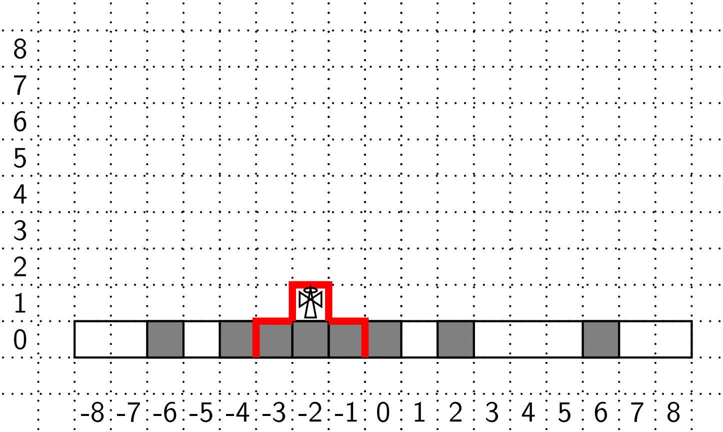
Catching a 1-Fool



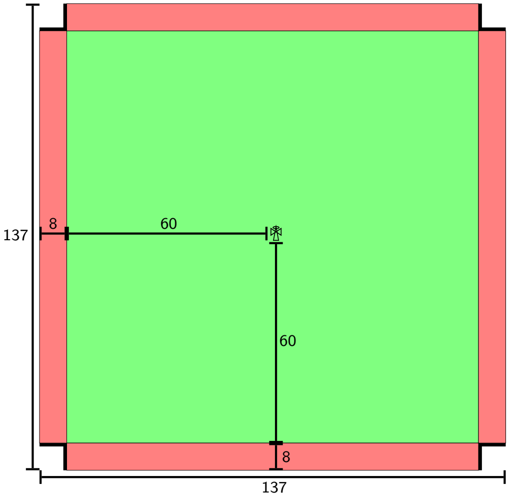
Catching a 1-Fool



Catching a 1-Fool



Catching a 1-Angel



Catching a 1-Angel

- We proved: a 1-Angel can be caught on a 137×137 chessboard.
- Berlekamp (1982): 1-Angel can be caught on a 33×33, but not on any smaller square board.

Can a k-Angel win for any k?

- Berlekamp, Conway, Guy (1982): **The Angel Problem**
  Can an Angel of sufficient power win?

Can a k-Angel win for any k?

- Conway (1996):
  - \$100 for a winning Angel proof
  - \$1000 for a winning Devil proof
- The Devil can never "make a mistake".

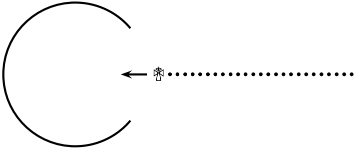
Scaring Angels into traps

Idea: use a *potential function* to tell where most eaten squares are, and run away from there.

Idea: use a *potential function* to tell where most eaten squares are, and run away from there.

Problem:

- Sensitive to nearby squares. . .

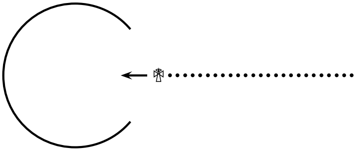


Scaring Angels into traps

Idea: use a *potential function* to tell where most eaten squares are, and run away from there.

Problem:

- Sensitive to nearby squares...



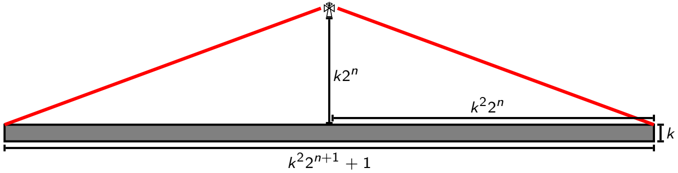
- Sensitive to faraway squares...

Catching a *k*-Fool

Catching a k-Fool

Like we caught the 1-Fool:

- Wall at distance $k\,2^n \implies k \cdot \left(k^2 2^{n+1} + 1\right)$ squares.



Catching a k-Fool

Like we caught the 1-Fool:

- Wall at distance $k\,2^n \implies k \cdot \left(k^2 2^{n+1} + 1\right)$ squares.
- Angel's distance halves $\implies$ Devil can eat 2^{n-1} squares $=$ 1 in $4\,k^3$.

Catching a k-Fool

Like we caught the 1-Fool:

- Wall at distance $k\,2^n \implies k \cdot \left(k^2 2^{n+1} + 1\right)$ squares.
- Angel's distance halves $\implies$ Devil can eat 2^{n-1} squares $= 1$ in $4\,k^3$.
- Angel can only reach half the width of the wall
  $\implies$ Devil only needs to consider half as many squares.

Catching a k-Fool

Like we caught the 1-Fool:

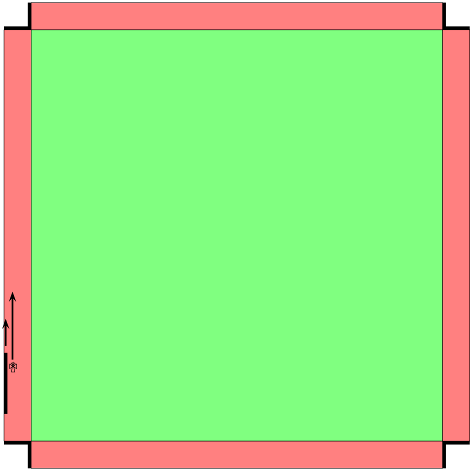
- Wall at distance $k\,2^n \implies k \cdot \left(k^2 2^{n+1} + 1\right)$ squares.
- Angel's distance halves $\implies$ Devil can eat 2^{n-1} squares $= 1$ in $4\,k^3$.
- Angel can only reach half the width of the wall
  $\implies$ Devil only needs to consider half as many squares.
- Angel's distance : 2 $\implies$ Devil can eat 1 in $4\,k^3$ squares.

Catching a k-Fool

Like we caught the 1-Fool:

- Wall at distance $k\,2^n \implies k \cdot \left(k^2 2^{n+1} + 1\right)$ squares.
- Angel's distance halves $\implies$ Devil can eat 2^{n-1} squares $= 1$ in $4\,k^3$.
- Angel can only reach half the width of the wall
  $\implies$ Devil only needs to consider half as many squares.
- Angel's distance : 2 $\implies$ Devil can eat 1 in $4\,k^3$ squares.
- Take $n = 4\,k^3$ (start to build wall at distance $k\,2^{4k^3}$)
  $\implies$ last hole in wall filled just before the Angel tries to break through.

The 1-Angel strategy for the Devil does not generalize



The Angel wins!

Suddenly in 2006: four independent proofs that some Angel wins:

- **Peter Gács**: Angel of some power wins

The Angel wins!

Suddenly in 2006: four independent proofs that some Angel wins:

- **Peter Gács**: Angel of some power wins
- **Brian Bowditch**: 4-Angel wins

The Angel wins!

Suddenly in 2006: four independent proofs that some Angel wins:

- **Peter Gács**: Angel of some power wins
- **Brian Bowditch**: 4-Angel wins
- **András Máthé**: 2-Angel wins
- **Oddvar Kloster**: 2-Angel wins

The Angel wins!

Suddenly in 2006: four independent proofs that some Angel wins:

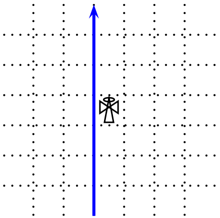
- **Peter Gács**: Angel of some power wins
- **Brian Bowditch**: 4-Angel wins
- **András Máthé**: 2-Angel wins
- **Oddvar Kloster**: 2-Angel wins

I present Oddvar Kloster's proof, because:

- It's a simple and insightful proof;
- It gives an explicit and easy-to-implement winning strategy for the 2-Angel.
- The Angel does not even need to fly.

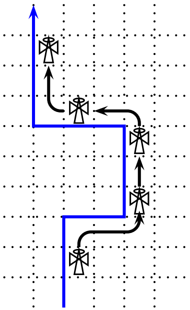
Oddvar Kloster's proof

- The Angel walks along the right of a directed **path**
- Initially the path a vertical line, with the angel directly to the right of it



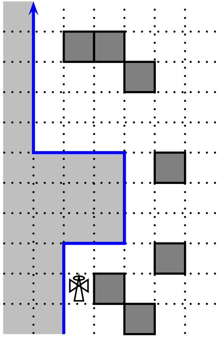
Oddvar Kloster's proof

- Each turn, the Angel advances as far along the path as possible.
- The Angel moves along at least 2 segments of the path each time.
- If he makes a right turn, the Angel moves along more than 2 segments in one go.



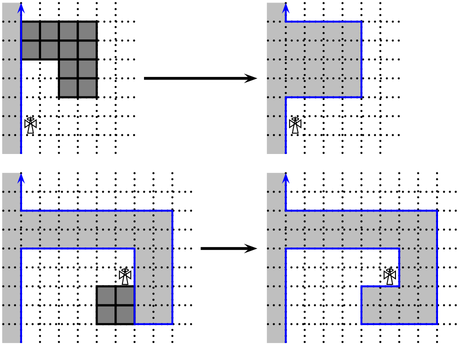
Oddvar Kloster's proof

- Squares are either **free** □ or **eaten** ■.
- Squares to the **left** of the path are called **evaded** □.



Oddvar Kloster's proof

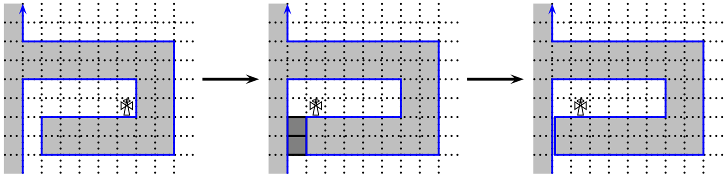
- Before moving, the Angel must adjust the path ahead, to avoid the Devil's traps.
- He is only allowed to **move some section of the path to the right** (w.r.t. the direction of the path) to evade some more squares:



- The resulting path is called a **decendant** of the original path.

Oddvar Kloster's proof

- The Angel has to be careful how he adjusts the path, to avoid being trapped:

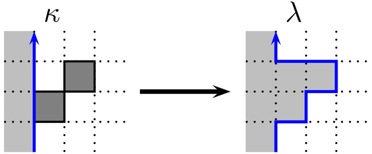


- The Angel needs a strategy that keeps the squares along the right of the path free and unevaded.

Oddvar Kloster's proof

- Each path is infinite, but differs only on a finite section from the original vertical path.
- So we can assign an integer **length $\mathbf{L}_\kappa$** to a path κ, namely the number of extra segments it has compared to the original path.
- At some stage s of play and for some path κ, we define $\mathbf{E}_\kappa(\mathbf{s})$ to be the number of **squares, eaten before stage s, that are evaded by** κ.

At stage s:

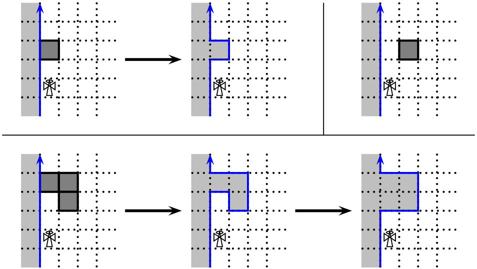


$$L_\lambda = L_\kappa + 4$$
$$E_\lambda(s) = E_\kappa(s) + 2$$

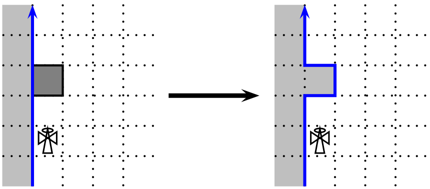
Oddvar Kloster's proof

How the Angel adjusts its path at stage s:

- Among all the decendants of the previous path, select the paths κ with maximal $2 \cdot E_\kappa(s) - L_\kappa$.
- Among these, select one with maximal $E_\kappa(s)$.
- Motto: *evade as many eaten squares as possible, taking into account a half penalty point for every increase in length.*

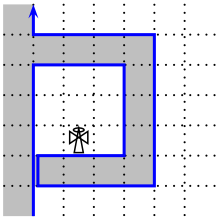


Suppose the Angels follows this strategy.
A square, just right of a future segment of the path, is not eaten.
Indeed the Angel will prefer the path that goes around it:



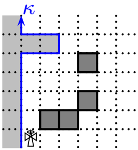
Oddvar Kloster's proof

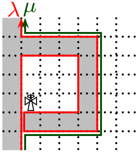
Can a square, just right of a future segment of the path, be evaded? If so, the Angel might be trapped:



We will prove that the Angel, following our strategy, would have spotted this potential trap in time, and would have planned its path around the trap.

Oddvar Kloster's proof

Stage s: 

Stage t: 

Angel passes segments more than twice faster than the Devil can eat:

$$L_\lambda - L_\mu > 2\,(E_\mu(t) - E_\mu(s))$$

So

$$\begin{aligned} 2 \cdot E_\mu(s) - L_\mu &> 2 \cdot E_\mu(t) - L_\lambda \\ &\geq 2 \cdot E_\lambda(t) - L_\lambda \\ &\geq 2 \cdot E_\kappa(s) - L_\kappa \end{aligned}$$

So at stage s the Angel would have preferred path μ over path κ.

In 3D

- Certainly an 2-Angel can win, by never changing its z-coordinate and applying Kloster's strategy in the plane.

In 3D

- Certainly an 2-Angel can win, by never changing its z-coordinate and applying Kloster's strategy in the plane.
- A 1-Angel can win in 3D, using only two consecutive z-coordinates:
  - Project down onto a plane.
  - A square eaten in space is a square half-eaten in the plane.
  - Half-eaten squares are still available for the angel.
  - Kloster's proof still works.

In 3D

- Certainly an 2-Angel can win, by never changing its z-coordinate and applying Kloster's strategy in the plane.
- A 1-Angel can win in 3D, using only two consecutive z-coordinates:
  - Project down onto a plane.
  - A square eaten in space is a square half-eaten in the plane.
  - Half-eaten squares are still available for the angel.
  - Kloster's proof still works.
- Open question: can a 1-Angel that always increases its z-coordinate win in 3D?
  - Bollobás and Leader (2006): Devil cannot win by building a wall at a fixed z-coordinate.

Further reading

- Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways for your mathematical plays*. (1982)
- Béla Bollobás and Imre Leader. *The angel and the devil in three dimensions*. (2006)
- Brian H. Bowditch, *The angel game in the plane*. (2007)
- John H. Conway. *The angel problem*. (1996)
- Peter Gács. *The angel wins*. (2007)
- Oddvar Kloster. *A Solution to the Angel Problem*. (2007)
- Oddvar Kloster. `http://home.broadpark.no/~oddvark/angel/`
- Martin Kutz. *The Angel Problem, Positional Games, and Digraph Roots*. (2004)
- András Máthé. *The angel of power* 2 *wins*. (2007)

Slides available on `http://www.stijnvermeeren.be/mathematics`