Realizability Toposes

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April 30, 2009

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Signed: ....

In the 1940s, Stephen Cole Kleene [4] discovered the realizability interpretation of intuitionistic number theory. In 1982, Martin Hyland [1] presented the effective topos, in which Kleene's realizability reappears. The aim of this essay is to explain these concepts, and to prove their connection.

In the first section I present Kleene's realizability. Sections 2 and 3 give a detailed presentation of the effective topos. In section 4, I show how Kleene's realizability reappears inside the effective topos. Finally, I briefly discuss some more advanced properties of the effective topos, and some ways to generalize its construction.

This essay was written as part of the Certificate of Advanced Study in Mathematics, commonly known as "*Part III*", at the University of Cambridge. The essay topic was proposed and supervised by Peter T. Johnstone.

My most important guides in writing this paper were the original articles of Kleene [4] and Hyland [1]. Other helpful resources were [10], [3], [2], [6] and the lecture notes from the Part III courses on Category Theory, lectured by Richard Garner, and on Topos Theory, lectured by Peter T. Johnstone.

Notation. We fix a Gödel numbering of the recursive functions. If $e, n \in \mathbb{N}$, we simply write $e(n)$ when we mean the result of applying the recursive function with number e to the input n. We also fix a recursive pairing function, and we write $\langle n, m \rangle$ for the encoding of the pair (n, m).

1 Kleene's realizability

Stephen Coole Kleene was in 1945 [4] the first to introduce the notion of realizability. As he relates in [5], Kleene was struck by the fact that both intuitionism and the theory of recursive functions deal with effective or constructive processes. Realizability is the result of Kleene's attempt to find a precise connection between intuitionistic number theory and recursive functions.

For an intuitionist, the statement $\exists x \varphi(x)$ only holds if we explicitly know a particular x for which $\varphi(x)$ holds. Such a value x can be said to *realize* the statement $\exists x \varphi(x)$. Kleene generalized this notion to arbitrary sentences in first order logic.

Definition 1.1. Consider some first order predicate language, with function and relation symbols corresponding to total recursive functions and total recursive relations on $\mathbb{N}$. We write $\overline{n}$ for the constant symbol corresponding to the constant function with value n.

A natural number e **realizes** a sentence φ if

(a) φ is an atomic formula, which is true when evaluating the corresponding recursive functions, and $e = 0$,

(b) $\varphi = \psi \wedge \chi$ and $e = \langle m, n \rangle$ where m realizes ψ and n realizes χ,

(c) $\varphi = \psi \vee \chi$ and either $e = \langle 0, m \rangle$ where m realizes ψ, or $e = \langle 1, n \rangle$ and n realizes χ,

(d) $\varphi = \psi \to \chi$ and whenever n realizes φ, then $e(n)$ is defined and realizes χ,

(e) $\varphi = \neg\psi$ and e realizes $\psi \to (\overline{0} = \overline{1})$,

(f) $\varphi = \exists x(\psi(x))$ and $e = \langle m, n \rangle$ where m realizes $\psi(\overline{n})$, or

(g) $\varphi = \forall x(\psi(x))$ and $e(n)$ realizes $\psi(\overline{n})$ for all $n \in \mathbb{N}$.

The main importance of realizability is providing an interpretation of intuitionistic number theory (also called Heyting arithmetic). Indeed, David Nelson [7] proved that if a sentence is provable in intuitionistic number theory, then it is realizable.

We give some examples of realizability:

- We look for an $e \in \mathbb{N}$ realizing

  $$\forall x(\exists y(2 \cdot y = x) \vee \neg\exists y(2 \cdot y = x)).$$

  By (g), $e(n)$ must realize

  $$\exists y(2 \cdot y = \overline{n}) \vee \neg\exists y(2 \cdot y = \overline{n}) \tag{1}$$

  for all $n \in \mathbb{N}$.

  If <u>n is even</u>, then by (a),

  $$2 \cdot \overline{\left(\frac{n}{2}\right)} = \overline{n}$$

  is realized by 0. So by (f),

  $$\exists y(2 \cdot y = \overline{n})$$

  is realized by $\left\langle 0, \frac{n}{2} \right\rangle$. Hence by (c), $\left\langle 0, \left\langle 0, \frac{n}{2} \right\rangle \right\rangle$ realizes (1).

  If <u>n is odd</u>, then there is no $m \in \mathbb{N}$ for which

  $$2 \cdot \overline{m} = \overline{n}$$

  is realized. So

  $$\exists y(2 \cdot y = \overline{n})$$

is not realized. Hence, by (d)

$$\exists y(2 \cdot y = \overline{n}) \rightarrow (\overline{0} = \overline{1})$$

is realized by any natural number, and using (e), so is

$$\neg \exists y(2 \cdot y = \overline{n}).$$

Hence $\langle 1, m \rangle$ realizes (1) for any $m \in \mathbb{N}$.

Consequently, e must be a recursive function that maps even numbers n to $\left\langle 0, \left\langle 0, \frac{n}{2} \right\rangle \right\rangle$, and maps odd numbers n to $\langle 1, f(n) \rangle$ for some arbitrary natural number $f(n)$.

- Let φ be any sentence. Then either φ is realized by some number, or any number realizes $\neg\varphi$. Hence

$$\varphi \vee \neg\varphi$$

is always realized.

- However, if $\varphi(x)$ is a formula with a free variable x, then

$$\forall x(\varphi(x) \vee \neg\varphi(x))$$

is only realized if there is an algorithm that tells, for each $n \in \mathbb{N}$, which of $\varphi(\overline{n})$ and $\neg\varphi(\overline{n})$ is realized.

- In particular (see [4] section 9), let $T(x, t)$ be the primitive recursive predicate "*the Turing machine with number x halts on input x after less than t steps*". Then any number realizing

$$\varphi = \forall x(\exists t T(x, t) \vee \neg \exists t T(x, t))$$

would correspond to a recursive solution of the haling problem, which is impossible. So we have a formula that is valid classically, but is not realized. Consequently the negation $\neg\varphi$ is realized, but invalid classically.

2 $\mathcal{P}(\mathbb{N})$-valued predicates

Kleene's realizability associates with each sentence φ a set of natural numbers that realize φ . We can think of this subset of $\mathbb{N}$ as a *nonstandard truth value* of φ .

Generalizing this, we define a $\mathcal{P}(\mathbb{N})$**-valued predicate** on a set X as a function $X \to \mathcal{P}(\mathbb{N})$. Such a $\mathcal{P}(\mathbb{N})$-valued predicate associates with each parameter in X a *nonstandard truth value*, that is a subset of $\mathbb{N}$.

For any set X, we have $\mathcal{P}(\mathbb{N})$-valued predicates $\top, \bot \in \mathcal{P}(\mathbb{N})^X$ given by

$$\begin{aligned} \top(x) &= \mathbb{N} \\ \bot(x) &= \emptyset \end{aligned}$$

for all $x \in X$.

Given $\varphi, \psi \in \mathcal{P}(\mathbb{N})^X$, we can define new $\mathcal{P}(\mathbb{N})$-valued predicates

$$(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi) \in \mathcal{P}(\mathbb{N})^X$$

by applying Kleene's definitions pointwise:

$$\begin{aligned} (\varphi \wedge \psi)(x) &= \{\ \langle n, m\rangle \ \mid\ n \in \varphi(x) \text{ and } m \in \psi(x)\ \} \\ (\varphi \vee \psi)(x) &= \{\ \langle 0, n\rangle \ \mid\ n \in \varphi(x)\ \} \cup \{\ \langle 1, m\rangle \ \mid\ m \in \psi(x)\ \} \\ (\varphi \to \psi)(x) &= \{\ e \in \mathbb{N} \ \mid\ \text{whenever } n \in \varphi(x) \text{, then } e(n) \in \psi(x)\ \} \end{aligned}$$

for all $x \in X$.

Next we can define a binary *entailment* relation $\vdash_X$ on $\mathcal{P}(\mathbb{N})^X$ by

$$\varphi \vdash_X \psi \ \textbf{ iff } \bigcap_{x \in X} (\varphi \to \psi)(x) \text{ is nonempty.}$$

Any element of $\cap_{x\in X}(\varphi \to \psi)(x)$ is a recursive function that sends each element of $\varphi(x)$ to an element of $\psi(x)$, regardless of the parameter $x \in X$. We say that such a function **uniformly realizes** $\varphi \vdash_X \psi$.

Proposition 2.1. *For each set X, $\left(\mathcal{P}(\mathbb{N})^X, \vdash_X\right)$ is a Heyting prealgebra.*

Proof. $\vdash_X$ is a <u>preorder</u> on $\mathcal{P}(\mathbb{N})^X$. Indeed the identity function uniformly realizes $\varphi \vdash_X \varphi$ for each $\varphi \in \mathcal{P}(\mathbb{N})^X$. And if f and g uniformly realize respectively $\varphi \vdash_X \psi$ and $\psi \vdash_X \chi$, then their composite $g \circ f$ uniformly realizes $\varphi \vdash_X \chi$.

$\top$ is a <u>maximum</u> for $\vdash_X$, because any total recursive function uniformly realizes $\varphi \vdash_X \top$ for any $\varphi \in \mathcal{P}(\mathbb{N})^X$. Similarly $\bot$ is a <u>minimum</u> for $\vdash_X$, as any recursive function uniformly realizes $\bot \vdash_X \varphi$ for any $\varphi \in \mathcal{P}(\mathbb{N})^X$.

We claim that $\varphi \wedge \psi$ is the <u>meet</u> of any $\varphi, \psi \in \mathcal{P}(\mathbb{N})^X$. Indeed $(\varphi \wedge \psi) \vdash_X \varphi$ is uniformly realized by projecting onto the first component, and $(\varphi \wedge \psi) \vdash_X \psi$ by projecting onto the second component. Moreover, if $\chi \in \mathcal{P}(\mathbb{N})^X$ and f, g uniformly realize respectively $\chi \vdash_X \varphi$ and $\chi \vdash_X \psi$, then

$$\langle f, g\rangle : n \mapsto \langle f(n), g(n)\rangle$$

uniformly realizes $\chi \vdash_X (\varphi \wedge \psi)$, as required.

Similarly, we prove that $\varphi \vee \psi$ is the join of any $\varphi, \psi \in \mathcal{P}(\mathbb{N})^X$. Indeed $\varphi \vdash_X (\varphi \vee \psi)$ is uniformly realized by the recursive function $n \mapsto \langle 0, n\rangle$, and $\psi \vdash_X (\varphi \vee \psi)$ by $n \mapsto \langle 1, n\rangle$. Moreover, if $\chi \in \mathcal{P}(\mathbb{N})^X$ and f, g uniformly realize respectively $\varphi \vdash_X \chi$ and $\psi \vdash_X \chi$, then the recursive function

$$h : \langle i, n\rangle \mapsto \begin{cases} f(n) & \text{if } i = 0 \\ g(n) & \text{if } i = 1 \end{cases}$$

uniformly realizes $(\varphi \vee \psi) \vdash_X \chi$.

Finally we show that $\varphi \to \psi$ is the Heyting implication of $\varphi, \psi \in \mathcal{P}(\mathbb{N})^X$. Suppose that $(\chi \wedge \varphi) \vdash_X \psi$ is uniformly realized by f. Define a recursive function

$$g : n \mapsto \text{Gödel number of the recursive function } f(\langle n, \cdot\rangle).$$

Then, whenever $x \in X$ and $n \in \chi(n)$, $g(n)$ maps each element of $\varphi(x)$ to an element of $\psi(x)$, that is $g(n) \in (\varphi \to \psi)(x)$. Hence g uniformly realizes $\chi \vdash_X (\varphi \to \psi)$.

Conversely, if $\chi \vdash_X (\varphi \to \psi)$ is uniformly realized by g, then define a recursive function

$$f : \langle n, m\rangle \mapsto (g(n))\,(m).$$

Then, whenever $x \in X$, $n \in \chi(x)$ and $m \in \varphi(x)$, we have $f(\langle n, m\rangle) \in \psi(x)$. So f uniformly realizes $(\chi \wedge \varphi) \vdash_X \psi$.

Thus we have proved $(\chi \wedge \varphi) \vdash_X \psi$ if and only if $\chi \vdash_X (\varphi \to \psi)$, as required. $\square$

Let $f : X \to Y$ be any function. Then we can define a map

$$f^* : \mathcal{P}(\mathbb{N})^Y \to \mathcal{P}(\mathbb{N})^X$$

by

$$(f^*\varphi)(x) = \varphi(f(x))$$

for all $\varphi \in \mathcal{P}(\mathbb{N})^Y$ and $x \in X$.

This map f^* is order-preserving, that is a functor

$$(\mathcal{P}(\mathbb{N})^Y, \vdash_Y) \to (\mathcal{P}(\mathbb{N})^X, \vdash_X).$$

Indeed, suppose $\varphi, \psi \in \mathcal{P}(\mathbb{N})^Y$, g uniformly realizes $\varphi \vdash_Y \psi$ and $x \in X$. Then g maps each element of $\varphi(f(x)) = (f^*\varphi)(x)$ to an element of $\psi(f(x)) = (f^*\psi)(x)$, so g also uniformly realizes $f^*\varphi \vdash_X f^*\psi$.

If f is surjective, then the converse (f^* is a full functor) also holds.

Now we define two maps $\forall f, \exists f : \mathcal{P}(\mathbb{N})^X \to \mathcal{P}(\mathbb{N})^Y$ by

$$(\exists f(\varphi))(y) = \bigcup_{x \in f^{-1}(\{y\})} \varphi(x)$$
$$(\forall f(\varphi))(y) = \{ \ e \in \mathbb{N} \ \mid \ e(0) \in A_y \ \}$$

where

$$A_y = \begin{cases} \bigcap_{x \in f^{-1}(\{y\})} \varphi(x) & \text{if } y \in \operatorname{im} f \\ \mathbb{N} \cup \{\text{undefined}\} & \text{otherwise} \end{cases}$$

for all $y \in Y$.

Proposition 2.2. *$\exists f$ and $\forall f$ are functors, respectively left and right adjoint to f^*.*

Proof. Suppose

$$\varphi \vdash_X \psi$$

is uniformly realized by g. Let $y \in Y$.

Any $n \in (\exists f(\varphi))(y)$ is an element of $\varphi(x)$ for some $x \in f^{-1}(\{y\})$. So $g(n) \in \psi(x)$ for the same $x \in f^{-1}(\{y\})$, that is $g(n) \in (\exists f(\psi))(y)$. Hence g also uniformly realizes

$$\exists f(\varphi) \vdash_Y \exists f(\psi),$$

so $\exists f$ is a functor.

Let g' be the recursive function that maps each recursive function e to $g \circ e$. If $e \in (\forall f(\varphi))(y)$, then $e(0) \in \varphi(x)$ for all $x \in f^{-1}(\{y\})$. So $(g'(e))(0) = g(e(0)) \in \psi(x)$ for all $x \in f^{-1}(\{y\})$, that is $g'(e) \in (\forall f(\psi))(y)$. Hence g' uniformly realizes

$$\forall f(\varphi) \vdash_Y \forall f(\psi),$$

so $\forall f$ is a functor.

Now let $\varphi \in \mathcal{P}(\mathbb{N})^X$ and $\psi \in \mathcal{P}(\mathbb{N})^Y$.

We first prove

$$\exists f(\varphi) \vdash_Y \psi \ \ \textbf{iff} \ \varphi \vdash_X f^*\psi.$$

Suppose $\exists f(\varphi) \vdash_Y \psi$ is uniformly realized by g. Then for any $x \in X$, if $n \in \varphi(x)$, also $n \in (\exists f(\varphi))(f(x))$, so $g(n) \in \psi(f(x)) = (f^*\psi)(x)$. Hence g also uniformly realizes $\varphi \vdash_X f^*\psi$.

Conversely, suppose g uniformly realizes $\varphi \vdash_X f^*\psi$. Then for each $y \in Y$, any $n \in (\exists f(\varphi))(y)$ is also element of $\varphi(x)$ for some $x \in f^{-1}(\{y\})$. So $g(n) \in (f^*\psi)(x) = \psi(y)$. Hence g uniformly realizes $\exists f(\varphi) \vdash_Y \psi$.

Hence $\exists f$ is left adjoint to f^*.

Next we prove

$$f^*\psi \vdash_X \varphi \ \textbf{ iff } \ \psi \vdash_Y \forall f(\varphi).$$

Suppose $f^*\psi \vdash_X \varphi$ is uniformly realized by g. Let g' be the recursive function that maps n to the constant function with value $g(n)$ (which is the everywhere undefined function if $g(n)$ is undefined). Then for any $y \in Y$, if $n \in \psi(y)$, then $n \in \psi(f(x)) = (f^*\psi)(x)$ for all $x \in f^{-1}(\{y\})$. So $g(n) \in \varphi(x)$ for all $x \in f^{-1}(\{y\})$. Hence $g'(n)$ is a recursive function with value $g(n) \in \varphi(x)$ for all $x \in f^{-1}(\{y\})$. So $g'(n) \in (\forall f(\varphi))(y)$ and g' uniformly realizes $\psi \vdash_Y \forall f(\varphi)$.

Conversely, suppose g uniformly realizes $\psi \vdash_Y \forall f(\varphi)$. Then let g' be the recursive function that maps n to $(g(n))(0)$. For any $x \in X$, if

$$n \in (f^*\psi)(x) = \psi(f(x)),$$

then $g(n) \in (\forall f(\varphi))(f(x))$. So

$$g'(n) = (g(n))(0) \in \varphi(x')$$

for all $x' \in f^{-1}(\{f(x)\})$, and in particular $g'(n) \in \varphi(x)$. So g' uniformly realizes $f^*\psi \vdash_X \varphi$.

Hence $\forall f$ is right adjoint to f^*. □

Remark 2.3. It is tempting to define

$$(\forall f(\varphi))(y) = \bigcap_{x \in f^{-1}(\{y\})} \varphi(x) \tag{2}$$

and claim that "*g uniformly realizes $f^*\psi \vdash_X \varphi$ if and only if g uniformly realizes $\psi \vdash_Y \forall f(\varphi)$*" for this definition of $\forall f$. However, this is <u>incorrect</u>. If g uniformly realizes $f^*\psi \vdash_X \varphi$ and $y \notin \operatorname{im} f$, then it is possible that there is some $n \in \psi(y)$ with $n \notin (f^*\psi)(x)$ for all $x \in X$. Then $g(n)$ might be undefined, which is impossible if g is to uniformly realize $\psi \vdash_Y \forall f(\varphi)$. This cannot be overcome by only allowing total recursive functions. For example the recursive function

$$f : \langle n, m \rangle \mapsto (g(n))(m)$$

that was needed in the proof of proposition 2.1 is not total. Only when f is surjective, (2) is a good definition of the right adjoint of f^*.

From now on, we will abuse our notation to some extent, and we will omit the subscript of the $\vdash$ symbol when it is clear what it should be. For example, suppose we have $\mathcal{P}(\mathbb{N})$-valued predicates $\varphi \in \mathcal{P}(\mathbb{N})^{X\times X}$ and $\psi \in \mathcal{P}(\mathbb{N})^X$. Then

$$\exists x'(\varphi(x, x')) \vdash \psi(x)$$

will mean that

$$\exists \pi_1(\varphi) \vdash_X \psi$$

where $\pi_1 : X \times X \to X$ is the projection onto the first component. Moreover

$$\varphi(x, x') \vdash \psi(x)$$

will mean that

$$\varphi \vdash_{X\times X} \pi_1^* \psi.$$

3 The effective topos

We are now ready to define the **effective topos Eff**.

3.1 Objects of Eff

An **object** of **Eff** is a set X together with a $\mathcal{P}(\mathbb{N})$**-valued equality predicate** $\approx \in \mathcal{P}(\mathbb{N})^{X\times X}$. This equality predicate must satisfy two axioms:

$$[x_1 \approx x_2] \vdash [x_2 \approx x_1] \qquad \text{(symmetry)}$$
$$[x_1 \approx x_2] \wedge [x_2 \approx x_3] \vdash [x_1 \approx x_3] \qquad \text{(transitivity)}$$

We do <u>not</u> require reflexivity. Indeed $[x \approx x]$ can even be the empty set for some $x \in X$. We call $[x \approx x]$ the **existence predicate** for $(X, \approx)$ and we abbreviate $[x \approx x]$ as $E(x)$.

For the name *existence predicate* to make sense, we certainly expect both sides of an *equality* to *exist*, that is

$$[x \approx x'] \vdash E(x) \wedge E(x'). \tag{3}$$

To prove this, we have to find a recursive function that, regardless of $x, x' \in X$, maps each element of $[x \approx x']$ to an element of $E(x) \wedge E(x')$. Suppose s uniformly realizes symmetry, and t uniformly realizes transitivity. If $n \in [x \approx x']$, then $s(n) \in [x' \approx x]$, so $t(\langle n, s(n)\rangle) \in [x \approx x]$ and $t(\langle s(n), n\rangle) \in [x' \approx x']$. Hence

$$n \mapsto \Big\langle\ t\big(\langle n, s(n)\rangle\big), t\big(\langle s(n), n\rangle\big)\ \Big\rangle$$

is a recursive function that uniformly realizes (3).

From now on, we will formulate our proofs in a more informal style. For example, we will proof (3) by saying: "*If* $[x \approx x']$*, then by symmetry also* $[x' \approx x]$*. Applying transitivity twice, we get* $[x \approx x]$ *and* $[x' \approx x']$*, and therefore* $E(x) \wedge E(x')$*.*" The fact that we can use this kind of intuitionistic reasoning with predicates to prove $\vdash$-relations, is called the Soundness Lemma in [2].

3.2 Morphisms of Eff

The **morphisms** $(X, \approx) \to (Y, \approx)$ of **Eff** are equivalence classes of functional relations $(X, \approx) \to (Y, \approx)$.

Here a **functional relation** $(X, \approx) \to (Y, \approx)$ is a $\mathcal{P}(\mathbb{N})$-valued predicate $F \in \mathcal{P}(\mathbb{N})^{X \times Y}$ such that the following axioms hold:

$$\begin{aligned} F(x,y) \wedge [x \approx x'] \wedge [y \approx y'] &\vdash F(x', y') && (F \text{ is relational}) \\ F(x,y) &\vdash E(x) \wedge E(y) && (F \text{ is strict}) \\ F(x,y) \wedge F(x,y') &\vdash [y \approx y'] && (F \text{ is single-valued}) \\ E(x) &\vdash \exists y (F(x,y)). && (F \text{ is total}) \end{aligned}$$

Two functional relations $F, G : (X, \approx) \to (Y, \approx)$ are **equivalent** if

$$F(x,y) \dashv\vdash G(x,y),$$

that is, both $F(x,y) \vdash G(x,y)$ and $G(x,y) \vdash F(x,y)$. This is indeed an equivalence relation on the functional relations, since $\vdash$ is a preorder on $\mathcal{P}(\mathbb{N})^{X \times Y}$ by proposition 2.1. Actually, $\vdash$ itself is already symmetric on functional relations:

Lemma 3.1. *If* F *and* G *are functional relations* $(X, \approx) \to (Y, \approx)$ *and* $F(x,y) \vdash G(x,y)$*, then also* $G(x,y) \vdash F(x,y)$*.*

Proof. Suppose $F(x,y) \vdash G(x,y)$. If $G(x,y)$, then strictness of G gives $E(x)$. By totality of F, $F(x,y')$ for some $y' \in Y$. By hypothesis, we infer that $G(x,y')$ for this $y' \in Y$. Since G is single-valued, we have $y \approx y'$. Finally, since F is relational, $F(x,y)$. So $G(x,y) \vdash F(x,y)$, as required. $\square$

If $G \in \mathcal{P}(\mathbb{N})^{X \times Y}$ is a function relation $(X, \approx) \to (Y, \approx)$, then $[\mathbf{G}]$ denotes its equivalence class, so $[G]$ is a morphism $(X, \approx) \to (Y, \approx)$.

If $f : (X, \approx) \to (Y, \approx)$ is a morphism, then $\mathbf{F_f} \in \mathcal{P}(\mathbb{N})^{X \times Y}$ denotes an arbitrary functional relation representing f.

We define the **composite** $g \circ f$ of two morphisms $f : (X, \approx) \to (Y, \approx)$ and $g : (Y, \approx) \to (Z, \approx)$ to be represented by

$$\begin{aligned} G : X \times Z &\to \mathcal{P}(\mathbb{N}) \\ (x, z) &\mapsto \exists y (F_f(x, y) \wedge F_g(y, z)). \end{aligned}$$

We first verify that G is a functional relation:

- G is relational: Suppose $G(x, z)$, $[x \approx x']$ and $[z \approx z']$. Then $F_f(x, y)$ and $F_g(y, z)$ for some $y \in Y$. As F_f is strict, $[y \approx y]$. Then by relationality of F_f and F_g we find $F_f(x', y)$ and $F_g(y, z')$. Hence $G(x', z')$ as required.
- G is strict: Immediate from the strictness of F_f and F_g.
- G is single-valued: Suppose $G(x, z)$ and $G(x, z')$. Then $F_f(x, y)$, $F_g(y, z)$, $F_f(x, y')$ and $F_g(y', z')$ for some $y, y' \in Y$. As F_f is single-valued, we have $[y \approx y']$. As F_g is strict, we have $[z \approx z]$. So by relationality of F_g, we get $F_g(y', z)$. Finally, as F_g is single-valued, $[z \approx z']$.
- G is total: Suppose $E(x)$. By totality of F_f we have $F_f(x, y)$ for some $y \in Y$, and $E(y)$ by strictness. By totality of F_g, we get $F_g(y, z)$ for some $z \in Z$. So we have $\exists y (F_f(x, y) \wedge F_g(y, z))$ for some $z \in Z$, that is $\exists z (G(x, z))$ as required.

We also need to check that $[G]$ is independent of the choice of representatives F_f of f and F_g of g. So suppose F'_f and F'_g are different representatives. They satisfy

$$\begin{aligned} F_f(x, y) &\dashv\vdash F'_f(x, y) \\ F_g(y, z) &\dashv\vdash F'_g(y, z). \end{aligned}$$

It immediately follows that

$$\exists y (F_f(x, y) \wedge F_g(y, z)) \dashv\vdash \exists y (F'_f(x, y) \wedge F'_g(y, z))$$

so the composite of f and g is indeed well-defined.

Proposition 3.2. Eff, *with objects and morphisms as above, is a category.*

Proof. We need to verify that composition is associative and we need to show that identity morphisms exist.

Associativity means that if $f : (V, \approx) \to (X, \approx)$, $g : (X, \approx) \to (Y, \approx)$ and $h : (Y, \approx) \to (Z, \approx)$, then

$$\begin{aligned} &\exists y \left(\exists x \left(F_f(v, x) \wedge F_g(x, y)\right) \wedge F_h(y, z)\right) \\ &\qquad\qquad \dashv\vdash \exists x \left(F_f(v, x) \wedge \exists y \left(F_g(x, y) \wedge F_h(y, z)\right)\right). \end{aligned}$$

This is obviously uniformly realized by the recursive *repairing* function

$$\langle\langle n,m\rangle, k\rangle \mapsto \langle n, \langle m,k\rangle\rangle$$

and its inverse.

Next we claim that $[\approx]$ is the identity morphism $id : (X,\approx) \to (X,\approx)$. The equality predicate $\approx$ is indeed a functional relation. It is relational, strict and single-valued by trivial application of symmetry and transitivity. (We have explicitly verified strictness as (3) above.) Totality is uniformly realized by the identity function.

Now, if $f : (Y,\approx) \to (X,\approx)$ and $g : (X,\approx) \to (Y,\approx)$. Then

$$\exists x'(G_f(y,x') \wedge [x' \approx x]) \dashv\vdash G_f(y,x)$$
$$\exists x'([x \approx x'] \wedge G_g(x',y)) \dashv\vdash G_f(x,y)$$

hold by strictness and relationality of G_f and G_g. So $[\approx] \circ f = f$ and $g \circ [\approx] = g$. Hence $[\approx] : (X,\approx) \to (X,\approx)$ is indeed the identity morphism of $(X,\approx)$. $\square$

We can give an easy description of isomorphisms in **Eff**.

Proposition 3.3. $f : (X,\approx) \to (Y,\approx)$ *is an isomorphism if and only if* F_f *is also a functional relation* $(Y,\approx) \to (X,\approx)$. *That is, if and only if we have*

$$F_f(x,y) \wedge F_f(x',y) \vdash [x \approx x'] \qquad (F_f \text{ is injective})$$
$$E(y) \vdash \exists x(F(x,y)) \qquad (F_f \text{ is surjective})$$

Proof. Suppose $f : (X,\approx) \to (Y,\approx)$ is an isomorphism. Then there is a $g : (Y,\approx) \to (X,\approx)$ such that $g \circ f = id_X$ and $f \circ g = id_Y$, that is

$$\exists y(F_f(x,y) \wedge F_g(y,x')) \dashv\vdash [x \approx x'] \tag{4}$$
$$\exists x(F_g(y,x) \wedge F_f(x,y')) \dashv\vdash [y \approx y']. \tag{5}$$

To prove injectivity, suppose $F_f(x,y)$ and $F_f(x',y)$. Then $E(y)$, so $F_g(y,x'')$ for some $x'' \in X$. But by applying (4) twice, we get $[x \approx x'']$ and $[x' \approx x'']$, so $[x \approx x']$.

To prove surjectivity, suppose $E(y)$. Then by (5) we immediately have $\exists x(F_f(x,y))$.

Conversely, suppose F_f is injective and surjective. Then $F_g(y,x) = F_f(x,y)$ is a functional relation $(Y,\approx) \to (X,\approx)$.

Suppose $F_f(x,y)$ and $F_g(y,x')$ (that is $F_f(x',y)$) for some $y \in Y$, then $[x \approx x']$ by injectivity. So (4) holds by lemma 3.1. Similarly, (5) holds by single-valuedness and lemma 3.1. Hence g is an inverse of f, and f is isomorphic. $\square$

3.3 Some limits in Eff

Proposition 3.4. $(\{*\}, \approx)$ *with* $[* \approx *] = \mathbb{N}$ *is a* ***terminal object*** *of* **Eff**.

Proof. Let $(X, \approx)$ be any object of **Eff**. We claim that the unique map

$$! : (X, \approx) \to (\{*\}, \approx)$$

is represented by

$$F(x, *) = E(x).$$

It is easy to check that this is indeed a functional relation $(X, \approx) \to (\{*\}, \approx)$. Let G be any other such functional relation. Then

$$G(x, *) \vdash E(x)$$

be strictness and

$$E(x) \vdash G(x, *)$$

by totality. So $[G] =!$, proving uniqueness. $\square$

Proposition 3.5. *The* ***product*** *of* $(X, \approx)$ *and* $(Y, \approx)$ *in* **Eff** *is* $(X \times Y, \approx)$ *where*

$$[(x, y) \approx (x', y')] = [x \approx x'] \wedge [y \approx y'],$$

with projections $\pi_1 : (X \times Y, \approx) \to (X, \approx)$ *and* $\pi_2 : (X \times Y, \approx) \to (Y, \approx)$ *represented by*

$$F_{\pi_1}((x, y), x') = [x \approx x'] \wedge E(y)$$
$$F_{\pi_2}((x, y), y') = E(x) \wedge [y \approx y'].$$

Proof. It is easy to check that everything in the proposition is well-defined.

If we are given morphisms $f : (A, \approx) \to (X, \approx)$ and $g : (A, \approx) \to (Y, \approx)$, then we define

$$< f, g >: (A, \approx) \to (X \times Y, \approx)$$

by

$$F_{<f,g>}(a, (x, y)) = F_f(a, x) \wedge F_g(a, y).$$

It is again easy to check that this $F_{<f,g>}$ is indeed a functional relation. Also we have

$$\exists(x', y)(F_f(a, x') \wedge G_g(a, y) \wedge [x' \approx x] \wedge E(y)) \dashv\vdash F_f(a, x)$$
$$\exists(x, y')(F_f(a, x) \wedge G_g(a, y') \wedge E(x) \wedge [y' \approx y]) \dashv\vdash F_g(a, y)$$

so we have

$$\pi_1 \circ < f, g >= f \quad \text{and} \quad \pi_2 \circ < f, g >= g. \tag{6}$$

Finally, any $k : (A, \approx) \to (X \times Y, \approx)$ with

$$\exists(x', y)(F_k(a, (x', y)) \wedge [x' \approx x] \wedge E(y)) \dashv\vdash F_f(a, x)$$
$$\exists(x, y')(F_k(a, (x, y')) \wedge E(x) \wedge [y' \approx y]) \dashv\vdash F_g(a, y)$$

must satisfy

$$F_k(a, (x, y)) \dashv\vdash F_f(a, x) \wedge F_g(a, y)$$

so $< f, g >$ is the unique morphism satisfying (6). $\square$

Proposition 3.6. *The* ***equalizer*** *of* $f, g : (X, \approx) \to (Y, \approx)$ *in* **Eff** *is*

$$e : (X, \approx_{eq}) \to (X, \approx)$$

with

$$[x \approx_{eq} x'] = [x \approx x'] \wedge \exists y(F_f(x, y) \wedge F_g(x, y))$$

and

$$F_e(x, x') = [x \approx_{eq} x'].$$

Proof. It is easily checked that $\approx_{eq}$ and F_e are well-defined. We need to show that $f \circ e = g \circ e$, that is

$$\exists x'([x \approx_{eq} x'] \wedge F_f(x', y)) \dashv\vdash \exists x'([x \approx_{eq} x'] \wedge F_g(x', y)).$$

We prove the $\vdash$ direction. (The converse is proven similarly or follows by lemma 3.1.) Suppose $[x \approx_{eq} x']$ and $F_f(x', y)$ for some $x' \in X$. By definition of $\approx_{eq}$ we have $F_f(x', y')$ and $F_g(x', y')$ for some $y' \in Y$. But since F_f is single-valued, we have $[y \approx y']$ and so $F_g(x', y)$. So we have $[x \approx_{eq} x']$ and $F_g(x', y)$ for some $x' \in X$, as required.

Suppose given $k : (V, \approx) \to (X, \approx)$ with $f \circ k = g \circ k$. We then claim that F_k is also a functional relation $(V, \approx) \to (X, \approx_{eq})$. Relationality and totality are trivial because $[x \approx_{eq} x'] \vdash [x \approx x']$. Strictness will follow trivially once we have proven single-valuedness. So suppose $F_k(v, x)$ and $F_k(v, x')$. We need to prove $[x \approx_{eq} x']$, but we are only given that $[x \approx x']$. However, by totality of F_f and F_g we find $y_f, y_g \in Y$ such that $F_f(x, y_f)$ and $F_g(x, y_g)$. As $f \circ k = g \circ k$, we deduce from $F_k(v, x)$ and $F_f(x, y_f)$ that $F_k(v, x'')$ and $F_g(x'', y_f)$ for some $x'' \in X$. But then we must have $[x \approx x'']$ and so $F_g(x, y_f)$. Hence we have $\exists y(F_f(x, y) \wedge F_g(x, y))$, so indeed $[x \approx_{eq} x']$.

We now obviously have

$$\exists x'(F_k(v, x') \wedge [x' \approx_{eq} x]) \dashv\vdash F_k(v, x)$$

so $k : (V, \approx) \to (X, \approx)$ factors through $e : (X, \approx_{eq}) \to (X, \approx)$.

Any other factorization $l : (V, \approx) \to (X, \approx_{eq})$ satisfies by definition

$$\exists x'(F_l(v, x') \wedge [x' \approx_{eq} x]) \dashv\vdash F_k(v, x).$$

But both F_l and F_k are functional relations $(V, \approx) \to (X, \approx_{eq})$, so we have

$$F_l(v, x) \dashv\vdash F_k(v, k).$$

Hence the factorization is unique. $\square$

Proposition 3.7. *A commutative square*

$$\begin{tikzcd}
(P, \approx) \arrow[r, "l"] \arrow[d, "k"'] & (Y, \approx) \arrow[d, "g"] \\
(X, \approx) \arrow[r, "f"'] & (Z, \approx)
\end{tikzcd} \tag{7}$$

is a ***pullback*** *in* **Eff** *if and only if*

$$< k, l >: (P, \approx) \to (X \times Y, \approx)$$

is injective and

$$F_f(x, z) \wedge F_g(y, z) \vdash \exists p(F_k(p, x) \wedge F_l(p, y)). \tag{8}$$

Proof. Suppose that (7) is a pullback. There is a canonical way of constructing the pullback, by taking the equalizer $e : (X \times Y, \approx_{eq}) \to (X \times Y, \approx)$ of $f \circ \pi_1, g \circ \pi_2 : (X \times Y, \approx) \to (Z, \approx)$. The limit in (7) must be isomorphic to this canonical limit, in the sense that there is an isomorphism $h : (P, \approx) \xrightarrow{\cong} (X \times Y, \approx_{eq})$ such that

$$\begin{tikzcd}
(P, \approx) \arrow[dr, "\cong", "h"'] \arrow[ddrrr, bend left, "l"] \arrow[dddrr, bend right, "k"'] & & & \\
& (X \times Y, \approx_{eq}) \arrow[dr, "e"] & & \\
& & (X \times Y, \approx) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (Y, \approx) \arrow[d, "g"] \\
& & (X, \approx) \arrow[r, "f"'] & (Z, \approx)
\end{tikzcd}$$

commutes.

To prove that

$$< k, l >= e \circ h : (P, \approx) \to (X \times Y, \approx)$$

is injective, suppose that we have $F_{<k,l>}(p, (x, y))$ and $F_{<k,l>}(p', (x, y))$. Then we have $F_h(p, (x_1, y_1))$, $F_e((x_1, y_1), (x, y))$ and $F_h(p', (x'_1, y'_1))$, $F_e((x'_1, y'_1), (x, y))$ for some $x_1, x'_1 \in X$ and $y_1, y'_1 \in Y$. By the definition of F_e from 3.6, we have $[(x_1, y_1) \approx_{eq} (x, y)]$ and $[(x'_1, y'_1) \approx_{eq} (x, y)]$. Hence we have both $F_h(p, (x, y))$ and $F_h(p', (x, y))$. Since h is an isomorphism, h is injective by proposition 3.3, so $[p \approx p']$ as required.

To prove (8), suppose that $F_f(x, z)$ and $F_g(y, z)$. Then $F_{f \circ \pi_1}((x, y), z)$ and $F_{g \circ \pi_2}((x, y), z)$ by the construction of the product in 3.5. So $F_e((x, y), (x, y))$ by the definition of equalizer in 3.6. Since h is an isomorphism, h is surjective by proposition 3.3, so $F_h(p, (x, y))$ for some $p \in P$. Consequently $F_k(p, x)$ and $F_l(p, y)$ for some $p \in P$, so $\exists p(F_k(p, x) \wedge F_l(p, y))$ as required.

Conversely, suppose that $< k, l >$ is injective and that (8) holds. To prove that (7) is a pullback, it is sufficient to show that factorization of (7) through the canonically constructed pullback is an isomorphism. By definition of product in 3.5 and equalizer in 3.6, this factorization is represented by $F_{<k,l>}$ as functional relation $(P, \approx) \to (X \times Y, \approx_{eq})$. By proposition 3.3, we only need to show that this functional relation is injective and surjective.

The injectivity of $F_{<k,l>}$ as functional relation $(P, \approx) \to (X \times Y, \approx_{eq})$ follows immediately from its injectivity as functional relation $(P, \approx) \to (X \times Y, \approx)$.

To prove surjectivity, suppose that $E_{eq}(x, y)$. Then by definition of $\approx_{eq}$, we have $F_f(x, z) \wedge F_g(y, z)$ for some $z \in Z$. By (8) we deduce that $F_k(p, x) \wedge F_l(p, y)$ for some $p \in P$. Thus, by definition of $F_{<k,l>}$, we have $F_{<k,l>}(p, (x, y))$ for this $p \in P$, as required. □

3.4 Monomorphisms and subobjects

As a corollary of the description of pullbacks above, we can characterize the monomorphisms of **Eff**.

Proposition 3.8. $f : (X, \approx) \to (Y, \approx)$ *is a **monomorphism** if and only if* F_f *is injective, that is*

$$F_f(x, y) \wedge F_f(x', y) \vdash [x \approx x'].$$

Proof. $f : (X, \approx) \to (Y, \approx)$ is a monomorphism if and only if

$$\begin{tikzcd}
(X,\approx) \arrow[r, "id"] \arrow[d, "id"'] & (X,\approx) \arrow[d, "f"] \\
(X,\approx) \arrow[r, "f"'] & (Y,\approx)
\end{tikzcd}$$

is a pullback. By the previous proposition, this is equivalent to

$$F_f(x, y) \wedge F_f(x', y) \vdash \exists x''([x'' \approx x] \wedge [x'' \approx x']).$$

But

$$\exists x''([x'' \approx x] \wedge [x'' \approx x']) \dashv\vdash [x \approx x'],$$

so the result follows. $\square$

Subobjects of $(X, \approx)$ can always be represented by monomorphisms of a canonical form:

Proposition 3.9. *Every subobject of* $(X, \approx)$ *can be represented by a monomorphism*

$$\iota_G : (X, \approx_G) \rightarrowtail (X, \approx)$$

where

$$[x \approx_G x'] = G(x) \wedge [x \approx x']$$

for some $G \in \mathcal{P}(\mathbb{N})^X$ *with*

$$[x \approx x'] \wedge G(x) \vdash G(x') \qquad (G \text{ is relational})$$

$$G(x) \vdash E(x) \qquad (G \text{ is strict})$$

and where

$$F_{\iota_G}(x, x') = [x \approx_G x'].$$

Proof. Given an injective functional relation $F_f : (Y, \approx) \rightarrowtail (X, \approx)$, we can define

$$G(x) = \exists y(F_f(y, x)).$$

This G is relational and strict, and $\iota_G : (X, \approx_G) \rightarrowtail (X, \approx)$ as defined in the proposition is a well-defined monomorphism. Moreover, F_f is also a functional relation $(Y, \approx) \to (X, \approx_G)$ which is injective and also (by definition of $\approx_G$) surjective. So F_f represents an isomorphism $k : (Y, \approx) \to (X, \approx_G)$. Finally,

$$\begin{tikzcd}
(Y,\approx) \arrow[rr, tail, "f"] \arrow[dr, tail, "k"', "\cong"] & & (X,\approx) \\
& (X,\approx_G) \arrow[ur, tail, "\iota_G"'] &
\end{tikzcd}$$

commutes, because

$$\exists x'(F_f(y, x') \wedge [x' \approx_G x]) \vdash F_f(y, x).$$

So ι_G represents the same subobject as f, as required. $\square$

Conversely, if $G \in \mathcal{P}(\mathbb{N})^X$ is relational (strictness is not necessary) then $\iota_G : (X, \approx_G) \rightarrowtail (X, \approx)$ as in the proposition above is always a well-defined monomorphism. We will denote the subobject represented by this ι_G as

$$G \rightarrowtail (X, \approx).$$

The subobjects of $(X, \approx)$ form a Heyting algebra, whose order relation is described by the following proposition.

Proposition 3.10. *If $G, H \in \mathcal{P}(\mathbb{N})^X$ are relational, then*

$$G \leq H$$

as subobjects of $(X, \approx)$, if and only if

$$G \wedge E \vdash_X H.$$

Proof. If $G \leq H$, then there is a monomorphism $f : G \rightarrowtail H$ such that

$$\begin{tikzcd} G \arrow[rr, tail, "\iota_G"] \arrow[dr, tail, "f"'] & & (X, \approx) \\ & H \arrow[ur, tail, "\iota_H"'] & \end{tikzcd} \tag{9}$$

commutes. Suppose $G(x) \wedge E(x)$. Then $F_{\iota_G}(x, x)$, so since the diagram commutes there is an $x' \in X$ with $F_{\iota_H}(x', x) = H(x') \wedge [x' \approx x]$. Because H is relational, we get $H(x)$.

Conversely, if $G(x) \wedge E(x) \vdash H(x)$, then

$$F_f(x, x') = F_{\iota_G}(x, x') = G(x) \wedge [x \approx x']$$

is also an injective functional relation $G \rightarrowtail H$ (it's strict per assumption), making (9) commute. So $G \leq H$. $\square$

Remark 3.11. If we require that G and H in the proposition are also strict (which is not a significant restriction by proposition 3.9), then $G \leq H$ if and only if $G \vdash_X H$. This is the way in which the proposition is usually presented in the literature (for example [1] and [10]). However, we will allow subobjects of $(X, \approx)$ also to be represented by elements of $\mathcal{P}(\mathbb{N})^X$ that are only relational and not strict. This approach slightly simplifies in propositions 3.12 and 4.3.

Proposition 3.12. *The Heyting algebra structure of* $Sub(X,\approx)$ *corresponds to the Heyting algebra structure of* $(\mathcal{P}(\mathbb{N})^X, \vdash_X)$.

Proof. Suppose $G, H \rightarrowtail (X,\approx)$ are two subobjects, represented by $G, H \in \mathcal{P}(\mathbb{N})^X$.

We claim that the meet of G and H in $Sub(X,\approx)$ is represented by $(G \wedge H) \in \mathcal{P}(\mathbb{N})^X$. Indeed, $G \wedge H$ is relational if G and H are, so it represents a subobject. By the proposition above we have $G \wedge H \leq G$ and $G \wedge H \leq H$. And if $F \in \mathcal{P}(\mathbb{N})^X$ satisfies $F \wedge E \vdash_X G$ and $F \wedge E \vdash_X H$, then

$$F \wedge E \vdash_X G \wedge H$$

by definition of join in $(\mathcal{P}(\mathbb{N})^X, \vdash)$. Hence $F \leq G$ and $F \leq H$ implies $F \leq G \wedge H$, as required.

Similarly, the join of G and H in $Sub(X,\approx)$ is represented by $(G \vee H) \in \mathcal{P}(\mathbb{N})^X$.

Finally we claim that the Heyting implication of G and H in $Sub(X,\approx)$ is represented by $(G \to H) \in \mathcal{P}(\mathbb{N})^X$. Indeed, $G \to H$ is relational if G and H are, so it represents a subobject. And $F \wedge G \leq H$ if and only if $F \wedge G \wedge E \vdash_X H$, if and only if $F \wedge E \vdash_X (G \to H)$, if and only if $F \leq (G \to H)$. □

3.5 It's a topos

Theorem 3.13. Eff *is a topos.*

Proof. We've already proved that **Eff** has a terminal object, binary products and equalizers. Hence **Eff** has all finite limits. So we only need to prove that **Eff** has a subobject classifier and exponentials.

For the subobject classifier, take

$$\Omega = (\mathcal{P}(\mathbb{N}), \approx)$$

with

$$[A \approx B] = (A \to B) \wedge (B \to A).$$

Hence an element of $[A \approx B]$ is a pair of two recursive funtions, one mapping elements of A to elements of B and the other vice versa. The map $\top : 1 \rightarrowtail \Omega$ is represented by

$$F_\top(*, A) = [A \approx \mathbb{N}].$$

Then, given a subobject $R \rightarrowtail (X,\approx)$, the unique morphism $\chi_R : (X,\approx) \to \Omega$ corresponding to it, is represented by

$$F_{\chi_R}(x, A) = E(x) \wedge [R(x) \approx A].$$

That this makes

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "\top"] \\ (X,\approx) \arrow[r, "\chi_R"] & (\mathcal{P}(\mathbb{N}),\approx) \end{tikzcd} \tag{10}$$

a pullback is easily verified, using the description of pullbacks in proposition 3.7. The only trick needed, is the fact that for all $A \subseteq \mathbb{N}$,

$$A \dashv\vdash [\mathbb{N} \approx A].$$

Indeed, given $n \in A$, we can form the constant function with value n to find an element of $[\mathbb{N} \approx A]$. And given an element of $[\mathbb{N} \approx A]$, we have a recursive function mapping elements of $\mathbb{N}$ to elements of A, and hence we can find an element of A.

To prove uniqueness of χ_R, suppose that F is any functional relation $(X,\approx) \to \Omega$ making (10) a pullback. We need to prove that

$$F(x, A) \vdash F_{\chi_R}(x, A).$$

By definition of χ_R is sufficient to prove that

$$F(x, A) \vdash (A \to R(x)) \wedge (R(x) \to A),$$

and using the Heyting algebra structure of proposition 2.1, it suffices to prove

$$\begin{aligned} F(x, A) \wedge A &\vdash R(x) \\ F(x, A) \wedge R(x) &\vdash A. \end{aligned}$$

This follows easily, again using proposition 3.7.

The underlying set of the <u>exponential</u> $(Y,\approx)^{(X,\approx)}$ of two objects $(X,\approx)$ and $(Y,\approx)$, is $\mathcal{P}(\mathbb{N})^{X\times Y}$, the set of all $\mathcal{P}(\mathbb{N})$-valued predicates on $X \times Y$.

Next define a $\mathcal{P}(\mathbb{N})$-valued predicate E on $\mathcal{P}(\mathbb{N})^{X\times Y}$, expressing that some $\mathcal{P}(\mathbb{N})$-valued predicate is a functional relation, by

$$\begin{aligned} E(F) = {} & \forall x, x', y, y'(F(x, y) \wedge [x \approx x'] \wedge [y \approx y'] \to F(x', y')) \\ & \wedge\ \forall x, y(F(x, y) \to E(x) \wedge E(y)) \\ & \wedge\ \forall x, y, y'(F(x, y) \wedge F(x, y') \to [y \approx y']) \\ & \wedge\ \forall x(E(x) \to \exists y(F(x, y))). \end{aligned}$$

Then we can define the equality predicate of the exponential $(Y,\approx)^{(X,\approx)}$ by

$$\begin{aligned} [F \approx G] = {} & E(F) \wedge E(G) \\ & \wedge\ \forall x, y(F(x, y) \to G(x, y)) \\ & \wedge\ \forall x, y(G(x, y) \to F(x, y)). \end{aligned}$$

We will not write out the straightforward but tedious verification that this in indeed an exponential. $\square$

4 The natural number object of Eff

A **natural number object** of a category is an object X together with morphisms $0 : 1 \to X$ and $s : X \to X$, such that for every object Y with morphisms $0' : 1 \to Y$ and $s' : Y \to Y$, there is a unique morphism $f : X \to Y$ such that

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "0'"'] & X \arrow[r, "s"] \arrow[d, dotted, "f"] & X \arrow[d, dotted, "f"] \\ & Y \arrow[r, "s'"] & Y \end{tikzcd}$$

commutes.

For example, in the category of sets, the set of natural numbers with zero and successor functions is a natural number object.

Proposition 4.1. **Eff** *has a natural number object* $(\mathbb{N}, \approx)$*, where*

$$[n \approx m] = \{n\} \cap \{m\},$$

with $0 : 1 \to (\mathbb{N}, \approx)$ *represented by*

$$F_0(*, n) = \{0\} \cap \{n\}$$

and with $s : (\mathbb{N}, \approx) \to (\mathbb{N}, \approx)$ *represented by*

$$F_s(n, m) = \{n + 1\} \cap \{m\}.$$

Proof. It is easy to check that everything in the proposition is well-defined.

Suppose we have an object $(Y, \approx) \in$ **Eff** and morphisms $0' : 1 \to (Y, \approx)$ and $s' : (Y, \approx) \to (Y, \approx)$.

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "0'"'] & (\mathbb{N}, \approx) \arrow[r, "s"] \arrow[d, dotted, "f"] & (\mathbb{N}, \approx) \arrow[d, dotted, "f"] \\ & (Y, \approx) \arrow[r, "s'"] & (Y, \approx) \end{tikzcd} \tag{11}$$

Let $F_f \in \mathcal{P}(\mathbb{N})^{\mathbb{N} \times Y}$ be defined by

$$F_f(n, y) = E(n) \wedge \exists y' \left(F_{0'}(*, y') \wedge F_{s'^n}(y', y) \right).$$

That F_f is a functional relation $(\mathbb{N}, \approx) \to (Y, \approx)$ follows immediately from the fact that $F_{0'}$ and $F_{s'}$ are functional relations.

To prove that (11) commutes, it is by lemma 3.1 sufficient to prove that

$$F_{0'}(*, y) \vdash \exists n(F_0(*, n) \wedge F_f(n, y)) \tag{12}$$

$$\exists y'(F_f(n, y') \wedge F_{s'}(y', y)) \vdash \exists n'(F_s(n, n') \wedge F_f(n', y)). \tag{13}$$

To prove (12), suppose that $F_{0'}(*, y)$. Then by definition of F_f we immediately find $F_f(0, y)$, and obviously we also have $F_0(*, 0)$, as required.

To prove (13), suppose that $F_f(n, y')$ and $F_{s'}(y', y)$ for some $y' \in Y$. Then by definition of F_f we immediately find $F_f(n+1, y)$, and obviously we also have $F_s(n, n+1)$, as required.

Finally, we have to prove that f is the unique morphism such that (11) commutes. So suppose that $g : (\mathbb{N}, \approx) \to (Y, \approx)$ also makes (11) commute. We prove that

$$F_g(n, y) \vdash F_f(n, y).$$

Indeed, given $F_g(n, y)$, we use strictness to find $n \in \mathbb{N}$. Then we can apply

$$\exists n'(F_s(n, n') \wedge F_g(n', y)) \vdash \exists y'(F_g(n, y') \wedge F_{s'}(y', y))$$

n times, and

$$\exists n(F_0(*, n) \wedge F_g(n, y)) \vdash F_{0'}(*, y)$$

once, to find

$$\exists y'(F_{0'}(*, y') \wedge F_{s'^n}(y', y))$$

and hence $F_f(n, y)$. □

We are now almost ready to rediscover Kleene's realizability inside the effective topos. We will define a structure in **Eff**, such that a sentence is true in this structure if and only if it's realizable in Kleene's sense.

For any first order language $\mathcal{L}$, an **$\mathcal{L}$-structure** of a topos is an object X of the topos, with for every l-ary function symbol f a morphism $f : X^l \to X$, and for every l-ary relation symbol R a subobject $R \rightarrowtail X^l$.

Remember that in Kleene's definition of realizability, we worked with a first order language where the function and relation symbols correspond to total recursive functions and relations. Given such a language $\mathcal{L}$, we define the **standard $\mathcal{L}$-structure** in **Eff** to have as underlying object the natural number object $(\mathbb{N}, \approx)$, with for each l-ary function symbol f a morphism $f : (\mathbb{N}^l, \approx) \to (\mathbb{N}, \approx)$ defined by

$$F_f(\vec{n}, m) = E(\vec{n}) \wedge [f(\vec{n}) \approx m],$$

and for each k-ary relation symbol R a subobject

$$R' \rightarrowtail (\mathbb{N}^l, \approx)$$

where $R' \in \mathcal{P}(\mathbb{N})^{\mathbb{N}^l}$ is defined by

$$R'(\vec{n}) = \begin{cases} \{0\} & \text{if } R(\vec{n}) \text{ holds} \\ \emptyset & \text{otherwise.} \end{cases}$$

Recapitulate how terms and formulas are interpreted in an $\mathcal{L}$-structure. Every **term** t with free variables among $\vec{x} = x_1, \cdots, x_k$ is **interpreted** as a morphism

$$X^k \xrightarrow{[\![\vec{x}.t]\!]} X.$$

Intuitively, the interpretation maps each *value* for the free variables to the corresponding *value* of the term. The interpretation is defined inductively: a variable x_i is interpreted as the projection π_i, and a composite term $f(s_1, \cdots, s_l)$ is interpreted as

$$X^k \xrightarrow{([\![\vec{x}.s_1]\!], \cdots, [\![\vec{x}.s_l]\!])} X^l \xrightarrow{f} X.$$

Similarly, each formula φ with free variables among $\vec{x}$ is interpreted as a subobject

$$[\![\vec{x}.\varphi]\!] \rightarrowtail X^n.$$

Intuitively, the interpretation is the subobject of all *values* of the free variables for which the formula is true. The interpretation is again defined inductively. An atomic formula $R(t_1, \cdots, t_l)$ is interpreted by taking the pullback

$$\begin{tikzcd} {[\![\vec{x}.R(t_1, \cdots, t_l)]\!]} \arrow[r, tail] \arrow[d] & X^k \arrow[d, "{([\![\vec{x}.t_1]\!], \cdots, [\![\vec{x}.t_l]\!])}"] \\ R \arrow[r, tail] & X^l \end{tikzcd}$$

Quantifier-free formulas are interpreted according to the Heyting algebra structure of the subobjects of X.

For quantifiers $\exists y$ and $\forall y$, consider the functor

$$\pi^*_{\vec{x}} : Sub(X^k) \to Sub(X^{k+1})$$

whose assignation is given by pulling back along

$$\begin{aligned} \pi_{\vec{x}} : X^{k+1} &\to X^k \\ (\vec{x}, y) &\mapsto (\vec{x}). \end{aligned}$$

This functor $\pi_{\vec{x}}^*$ has left and right adjoints

$$\exists\pi_{\vec{x}},\ \forall\pi_{\vec{x}} : Sub(X^{k+1}) \to Sub(X^k).$$

The interpretation of $\exists y\psi$ is then given by

$$[\![\vec{x}.\exists y\psi]\!] = \exists\pi_{\vec{x}}[\![\vec{x}, y.\psi]\!]$$

and similarly for $\forall y\psi$.

The next two propositions describe how quantification works for the natural number object of **Eff**.

Proposition 4.2. *Suppose* $R \rightarrowtail (\mathbb{N}, \approx)^k$ *is a subobject, where* $R \in \mathcal{P}(\mathbb{N})^{\mathbb{N}^k}$. *Suppose* $f : \mathbb{N}^l \to \mathbb{N}^k$ *is a total recursive function. Define a morphism* $g : f^*R \to R$ *by*

$$F_g(\vec{n}, \vec{m}) = R(\vec{m}) \wedge E(\vec{n}) \wedge [f(\vec{n}) \approx \vec{m}]$$

for all $\vec{n} \in \mathbb{N}^l$ *and* $\vec{m} \in \mathbb{N}^k$. *Then*

$$\begin{tikzcd} f^*R \arrow[r, tail, "\iota_{f^*R}"] \arrow[d, "g"'] & (\mathbb{N}^l, \approx) \arrow[d, "f"] \\ R \arrow[r, tail, "\iota_R"'] & (\mathbb{N}^k, \approx) \end{tikzcd} \tag{14}$$

is a pullback.

Proof. First note that $[\vec{n} \approx \vec{n'}]$ is empty for distinct $\vec{n}$ and $\vec{n'}$. Hence every element of $\mathcal{P}(\mathbb{N})^{\mathbb{N}^k}$ and $\mathcal{P}(\mathbb{N})^{\mathbb{N}^l}$ is relational. Remember that $f^*R = (\mathbb{N}^l, \approx_{f^*R})$ where

$$\begin{aligned}[\vec{n} \approx_{f^*R} \vec{n'}] &= f^*R(\vec{n}) \wedge [\vec{n} \approx \vec{n'}] \\ &= R(f(\vec{n})) \wedge [\vec{n} \approx \vec{n'}].\end{aligned}$$

It is easy to verify that F_g is a functional relation $f^*R \to R$, where we use the fact that f is recursive to prove totality. It is equally easy to show that (14) commutes, and $< g, \iota_{f^*R} >$ is injective because ι_{f^*R} is. So by proposition 3.7 we only need to prove that

$$F_{\iota_R}(\vec{m}, \vec{m'}) \wedge F_f(\vec{n}, \vec{m'}) \vdash \exists\vec{n'}(F_{\iota_{f^*R}}(\vec{n'}, \vec{n}) \wedge F_g(\vec{n'}, \vec{m}))$$

which is easily verified by filling in the definitions of these functional relations, and by choosing $\vec{n'} = \vec{n}$. $\square$

Proposition 4.3. *Let f be as in the previous proposition. The left and right adjoints of $f^* : Sub(\mathbb{N}^k, \approx) \to Sub(\mathbb{N}^l, \approx)$ are given by respectively*

$$G \mapsto \exists f(G \wedge E) \tag{15}$$

and

$$G \mapsto \forall f(E \to G). \tag{16}$$

Proof. We first verify that these assignments are functors, that is they preserve the order on subobjects. So suppose $G \leq H$ as subobjects of $Sub(\mathbb{N}^l, \approx)$, that is

$$G \wedge E \vdash_{\mathbb{N}^l} H. \tag{17}$$

Then also $G \wedge E \vdash_{\mathbb{N}^l} H \wedge E$ and because $\exists f$ is order-preserving $\exists f(G \wedge E) \vdash_{\mathbb{N}^l} \exists f(H \wedge E)$. Hence also $\exists f(G \wedge E) \leq \exists f(H \wedge E)$.

Similarly, because $(\mathcal{P}(\mathbb{N})^{\mathbb{N}^l}, \vdash_{\mathbb{N}^l})$ is a Heyting algebra, we get $(E \to G) \vdash_{\mathbb{N}^l} (E \to H)$ from (17). Using the fact that $\forall f$ is order-preserving, we find $\forall f(E \to G) \leq \forall f(E \to H)$.

Now if $E(\vec{n})$ for some $\vec{n} \in \mathbb{N}^l$ with $f(\vec{n}) = \vec{m} \in \mathbb{N}^k$, then we can use the fact that f is a recursive function to get $E(\vec{m})$. Hence we have

$$\exists f(G \wedge E) \dashv\vdash_{\mathbb{N}^k} \exists f(G \wedge E) \wedge E.$$

For the same reason, we have

$$f^*(G) \wedge E \dashv\vdash_{\mathbb{N}^l} f^*(G \wedge E) \wedge E.$$

Hence the following are equivalent for $F \in \mathcal{P}(\mathbb{N})^{N^k}$ and $G \in \mathcal{P}(\mathbb{N})^{N^l}$:

$$\begin{gathered} G \leq f^*F \\ G \wedge E \vdash_{\mathbb{N}^l} f^*F \\ \exists f(G \wedge E) \vdash_{\mathbb{N}^k} H \\ \exists f(G \wedge E) \wedge E \vdash_{\mathbb{N}^k} H \\ \exists f(G \wedge E) \leq H \end{gathered}$$

so (15) is indeed a left adjoint of f^*.

Similarly, the following are equivalent:

$$\begin{gathered} f^*F \leq G \\ f^*F \wedge E \vdash_{\mathbb{N}^l} G \\ f^*(F \wedge E) \wedge E \vdash_{\mathbb{N}^l} G \\ f^*(F \wedge E) \vdash_{\mathbb{N}^l} E \to G \\ F \wedge E \vdash_{\mathbb{N}^k} \forall f(E \to G) \\ F \leq \forall f(E \to G) \end{gathered}$$

so (16) is indeed a right adjoint of f^*. $\square$

Now we can finally prove that the standard interpretation of formulas in **Eff** corresponds to Kleene's realizability.

Proposition 4.4. *For each formula φ, with free variables among $\vec{x} = x_1, \cdots, x_k$, define $[\varphi] \in \mathcal{P}(\mathbb{N})^{\mathbb{N}^k}$ by*

$$[\varphi](\vec{n}) = \{\ e \in \mathbb{N} \ \mid \ e \textit{ realizes } \varphi(\vec{n})\ \}.$$

Then the interpretation $[\![\vec{x}.\varphi]\!]$ of φ in the standard $\mathcal{L}$-structure is exactly $[\varphi] \rightarrowtail \mathbb{N}^k$.

Proof. We use induction on the structure of the formula φ. For atomic formulas, the proposition follows immediately from the definition of the standard $\mathcal{L}$-structure. The induction steps for $\wedge$, $\vee$ and $\rightarrow$ follows from proposition 3.12, and the fact that the Heyting algebra structure of $(\mathbb{N}^k, \vdash)$ corresponds pointwise to the Kleene's definition of realizability of $\wedge$, $\vee$ and $\rightarrow$. So we only need to check existential and universal quantifiers.

Suppose $\varphi = \exists y(\psi)$. By proposition 4.3, the interpretation of φ is given by

$$\exists y([\psi](\vec{m}, y) \wedge E(\vec{m}, y)).$$

But it is easy to show that

$$\exists y([\psi](\vec{m}, y) \wedge E(\vec{m}, y)) \wedge E(\vec{m}) \dashv\vdash \exists y([\psi](\vec{m}, y) \wedge \{y\}) \wedge E(\vec{m}),$$

so by proposition 3.10 this subobject is also represented by

$$\exists y([\psi](\vec{m}, y) \wedge \{y\}),$$

which is exactly $[\varphi](\vec{m})$ according to Kleene's definition.

Similarly, suppose $\varphi = \forall y(\psi)$. By proposition 4.3, the interpretation of φ is given by

$$\forall y(E(\vec{m}, y) \rightarrow [\psi](\vec{m}, y)).$$

But it is easy to show that

$$\forall y(E(\vec{m}, y) \rightarrow [\psi](\vec{m}, y)) \wedge E(\vec{m}) \dashv\vdash \forall y(\{y\} \rightarrow [\psi](\vec{m}, y)) \wedge E(\vec{m}),$$

so by proposition 3.10 this subobject is also represented by

$$\forall y(\{y\} \rightarrow [\psi](\vec{m}, y)),$$

which is exactly $[\varphi](\vec{m})$ according to Kleene's definition. $\square$

As an immediate corollary we have

Proposition 4.5. *A sentence φ can be realized in Kleene's sense if and only if it is true for the standard $\mathcal{L}$-structure in* **Eff**.

5 Further properties and generalizations

The discovery of Kleene's realizability inside the effective topos is not the only interesting thing happening in **Eff**. As Van Oosten [9] writes, in the effective topos "many bits of research, up to then unrelated, fell into their right place". For example, from the natural number object $(\mathbb{N}, \approx)$ one can construct a **real number object** in **Eff**, similarly to the construction of the ordinary real numbers by Dedekind cuts or by Cauchy sequences. This real number object is given by $(\mathbb{R}_{rec}, \approx)$ where $\mathbb{R}_{rec}$ is the set of recursive reals (that is, real numbers with recursive Cauchy sequences converging to them) and

$$n \in [r \approx r']$$

if and only if $r = r'$ and n encodes a Cauchy sequences converging to r. From there, one can develop analysis inside the effective topos. This analysis turns out to correspond to so-called *recursive analysis*, where statements such as "*every function from the reals to the reals is continuous*" are valid. More details about analysis inside **Eff** and other important properties of the effective topos can be found in the references.

Andy Pitts [2] discovered that the construction of the effective topos is a particular example of a more general construction: the tripos-to-topos construction. Let $\mathcal{C}$ be any category and let `Heytpre` be the category of Heyting prealgebras. A **$\mathcal{C}$-tripos** is a pseudofunctor $\mathcal{C}^{op} \to$ `Heytpre` that satisfies a number of extra axioms (see [10]). The effective topos arises from the Set-tripos given by

$$\begin{aligned} X &\mapsto (\mathcal{P}(\mathbb{N})^X, \vdash_X) \\ f &\mapsto f^*. \end{aligned}$$

The axioms of a tripos ensure that we can construct a topos from any tripos, in a way similar to the definition of the effective topos.

Another way to generalize the construction of the effective topos, is by considering **partial combinatory algebras** (**pca**s), sometimes also called **Schönfinkel algebras** after [8]. A pca generalizes the way in which we can apply natural numbers to other natural numbers by

$$m(n) = \text{output of the recursive function with number } m \text{ to input } n.$$

Indeed, a pca is a set Λ together with a partial *application* map $\Lambda \times \Lambda \to \Lambda$, such that there are $k, s \in \Lambda$ with

$$\begin{aligned} (ka)b &= a \\ ((sa)b)c &\asymp (ac)(bc) \end{aligned}$$

for all $a, b, c \in \Lambda$. Here $\asymp$ means that one side is defined if the other is, and in that case both sides are equal. The element $ka \in \Lambda$ obviously behaves as the constant function with value a. But by combining k and s we can also find elements of Λ that handle composition, evaluation, pairing and projection, etc. With all this we can develop a theory of $\mathcal{P}(\Lambda)$-valued predicates, just like the $\mathcal{P}(\mathbb{N})$-valued predicates that we discussed in section 2. In this way, every pca Λ gives rise to a Set-tripos given by

$$\begin{aligned} X &\mapsto (\mathcal{P}(\Lambda)^X, \vdash_X) \\ f &\mapsto f^*, \end{aligned}$$

from which we can construct a *realizability* topos.

The pca where the natural numbers encode recursive functions (and which gives rise to the effective topos) is called the first Kleene algebra. Similarly, for any non-recursive set $E \subseteq \mathbb{N}$ there is a pca where the natural numbers encode oracle machines with an oracle for E. The topos arising from such a pca is a sheaf subtopos of the effective topos. But there are also very different examples of pcas (see for example [10]).

A useful approach in studing pcas and the toposes that arise from them, is to consider the category $\mathbf{Ass}(\Lambda)$ of Λ-valued assemblies for a pca Λ. A **Λ-valued assembly** is a pair (X, φ) where X is a set and $\varphi \in (\mathcal{P}(\Lambda) \setminus \{\emptyset\})^X$. A morphism between Λ-valued assemblies (X, φ) and (Y, ψ) is a function $f : X \to Y$ which is tracked by some $a \in \Lambda$, which means that if $b \in \varphi(x)$ then $ab \in \psi(f(x))$. The realizability topos that arises from the pca Λ is then the same as the *effectivization* of the category $\mathbf{Ass}(\Lambda)$. More details can be found in [3].

6 Further reading

Martin Hyland's article [1] from 1982, the first article about the effective topos, is still a good place to look for more details, especially since the content of this essay corresponds fairly well to the first sections of Hyland's article. A more modern and more extensive exposition can be found in Van Oosten's [10], the first and so far only book on realizability. A nice historical overview can be found in [9].

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